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Similarity laws for turbulent boundary layers with suction or injection

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The concept of a law of the wall and a velocity defect law which are related to each other through a common velocity scale and a semilogarithmic velocity profile in the region where they overlap, can be applied successfully to turbulent boundary layers with suction or injection. For turbulent boundary layers at moderate suction rates the velocity scale is proportional to u_τ^2/v_0. For layers at arbitrary suction or blowing rates the velocity scale, which has been determined empirically, is proportional to $(u_\tau + 9v_0)$.

1. Introduction

Since the theory of turbulent boundary-layer flow is still rather incomplete, it is impossible to provide a quantitative description of the mean flow which does not contain a large degree of empiricism. In the present state of the art, much of the available information centres around two similarity laws: the 'law of the wall' and the 'velocity defect law'. The law of the wall is valid in the thin 'inner layer' next to the wall, whereas the velocity defect law describes the flow in the broad 'outer layer'.

In this paper similarity laws for turbulent boundary layers with suction or injection are given. The applicability of these laws will be determined by comparison with experimental data. The analysis is restricted to incompressible turbulent boundary layers in steady two-dimensional flow. Mass transfer occurs normal to the porous surface only. The main part of the theory covers boundary layers at 'moderate' suction rates $(0{\cdot}04 < -v_0/u_\tau < 0{\cdot}10)$. The similarity laws for these boundary layers are based on a generalization of the phenomenological description of turbulent boundary-layer flow as given by Clauser (1956), Rotta (1962), Townsend (1956), Coles (1956) and others. A central feature of this description is the semi-logarithmic mean velocity profile in the region of overlap between the wall law and the velocity defect law. It will be seen that this feature can be retained successfully for the description of sucked or blown turbulent boundary layers. The approach presented here therefore deviates from the one taken in almost all literature on the subject (Dorrance & Dore 1954; Rubesin 1954; Clarke, Menkes & Libby 1955; Mickley & Davis 1957; Black & Sarnecki 1958; Sarnecki 1959; Cornish 1960; Townsend 1961; Stevenson 1963*a*, *b*). In these papers, mixing-length theory is applied to turbulent boundary-layer flow with suction or injection. In the resulting expression for the mean velocity profile a squared logarithm occurs, so that it has been labelled the 'bi-logarithmic law'. In the author's opinion this law conflicts with the concept of related similarity laws for the two regions in turbulent boundary layers. In only one reference (Mickley & Smith 1963) is the subject investigated in a manner similar to the one presented here. In a comment on Mickley & Smith's paper, the author has summarized some of the results of his investigation (Tennekes 1964*a*).

It seems suitable to introduce the analysis by giving a short review of the pertinent features of the phenomenological description of turbulent boundary-layer flow along impermeable surfaces. The law of the wall and the velocity defect law are, respectively (Prandtl 1935; Coles 1956; Townsend 1956; Clauser 1956; Rotta 1962)

$$\frac{\overline{U}_1}{u_\tau} = f\left(\frac{x_2 u_\tau}{\nu}; \frac{k u_\tau}{\nu}\right), \tag{1}$$

$$\frac{\overline{U}_1 - U_0}{u_\tau} = g\left(\frac{x_2}{\delta}; \Pi\right). \tag{2}$$

In these equations, $\overline{U}_1$ is the mean velocity within the boundary layer, U_0 is the mainstream velocity, u_τ is the friction velocity ($\rho u_\tau^2 = \tau_0$), k is a length scale for the surface roughness and Π is defined by (Rotta 1962)

$$\Pi = \frac{\delta^*}{\tau_0}\frac{d\overline{P}}{dx_1}. \tag{3}$$

The velocity defect law is valid only for boundary layers with constant value of Π. These layers are called 'equilibrium layers'. Postulating that the law of the wall and the velocity defect law will overlap in a finite region of the boundary layer, it follows that the mean velocity profile is semi-logarithmic in this region. The formal derivation of this feature is due to Millikan (1938) and runs as follows. Differentiation of (1) and (2) with respect to x_2 yields, with some rearrangement,

$$\frac{x_2}{u_\tau}\frac{\partial \overline{U}_1}{\partial x_2} = \frac{u_\tau x_2}{\nu}\frac{\partial f}{\partial(x_2 u_\tau/\nu)}, \tag{4}$$

$$\frac{x_2}{u_\tau}\frac{\partial \overline{U}_1}{\partial x_2} = \frac{x_2}{\delta}\frac{\partial g}{\partial(x_2/\delta)}. \tag{5}$$

The expressions at the right-hand side of (4) and (5) should be equal to each other in the region of overlap. Since they have no variable in common, $x_2 u_\tau/\nu$ being formally independent of x_2/δ, they should be constant. This implies that

$$\frac{x_2}{u_\tau}\frac{\partial \overline{U}_1}{\partial x_2} = C_0. \tag{6}$$

Since the functions f and g have no common parameters, C_0 is a 'universal constant'. Its value has to be determined by experiment. Clauser (1954, 1956) uses $C_0 = 2{\cdot}44$; in this paper $C_0 = 2{\cdot}3$ will be used to obtain agreement with data of blown and sucked boundary layers.

Integration of (6) and subsequent non-dimensionalization according to the law of the wall and the velocity defect law respectively yield the well-known semi-logarithmic mean velocity profile

$$\overline{U}_1/u_\tau = C_0 \ln(x_2 u_\tau/\nu) + A_0(k u_\tau/\nu), \tag{7}$$

$$(\overline{U}_1 - U_0)/u_\tau = C_0 \ln(x_2/\delta) + B_0(\Pi). \tag{8}$$

We now return to (6), which may be interpreted as follows. The slope of the semi-logarithmic region of the velocity profile in a semi-logarithmic plot has the dimensions of a velocity, i.e.

$$x_2(\partial \overline{U}_1/\partial x_2) = \partial \overline{U}_1/\partial(\ln x_2) = w^*. \tag{9}$$

The slope w^* is called the 'logarithmic velocity scale' (Tennekes 1964*a*) and is defined by (9). Comparison of (6) and (9) shows that for boundary layers without suction or blowing the logarithmic velocity scale is proportional to the friction velocity

$$w^* = C_0 u_\tau = 2{\cdot}3 u_\tau. \tag{10}$$

Apparently, it is merely a formalism to introduce w^* for boundary-layer flow along impermeable walls. However, it will be seen that for sucked and blown layers the introduction of w^* is very useful.

In the law of the wall and in the velocity defect law the mean velocity $\overline{U}_1$ is non-dimensionalized by the same variable, i.e. u_τ. This common velocity scale (instead of u_τ, w^* may be used) is the variable that relates the similarity laws for the two distinct regions of a turbulent boundary layer. To quote Clauser: 'The primary variable which interconnects these two mechanisms (i.e. the outer and the inner layer) is the wall shear τ_0 or its equivalent, the friction velocity u_τ' (Clauser 1956, p. 27). For boundary layers with suction or injection, u_τ is not a suitable velocity scale, but the concept of a joint velocity scale for the two similarity laws will be retained. This is of direct relevance to the velocity profile, since without a joint velocity scale no region of overlap and hence no semi-logarithmic velocity profile in the region of overlap can exist. It will be seen in the sections to follow that these concepts (with w^* as the appropriate velocity scale) provide for a simple description of turbulent boundary-layer flow with suction or injection.

2. The law of the wall for turbulent boundary layers at moderate suction rates

For turbulent boundary layers with suction a practical way to derive a wall law is to consider the flow in the viscous sublayer. The mean velocity profile in the sublayer is obtained directly from the equations of motion. In the approximation used for the inner layer (Black & Sarnecki 1958) these equations reduce for the viscous sublayer to

$$v_0(\partial \overline{U}_1/\partial x_2) = \nu(\partial^2 \overline{U}_1/\partial x_2^2). \tag{11}$$

This equation may be integrated twice to obtain the velocity profile

$$v_0 \overline{U}_1/u_\tau^2 = \exp(v_0 x_2/\nu) - 1. \tag{12}$$

This is equivalent to the velocity distribution given by Griffith & Meredith (1936) for laminar asymptotic layers.

The velocity profiles in the viscous sublayers of all turbulent boundary layers with suction or injection apparently coincide if they are plotted as

$$v_0 \overline{U}_1/u_\tau^2 = F(v_0 x_2/\nu). \tag{13}$$

The similarity of the sublayer flow according to (13) is surprisingly universal since it is independent of the 'suction ratio' $-v_0/u_\tau$. Since a similarity law for the flow in the inner layer should at least provide similarity of the flow in the viscous sublayer (which is the lowermost part of the inner layer), (13) might be an appropriate law of the wall for boundary layers on a permeable surface.

Equation (13), which will be called the 'limit law of the wall' for reasons set out later, is not a suitable similarity law at other than moderate suction rates. This may be shown as follows. Expanding (12) into a series, we obtain

$$\frac{v_0 \overline{U}_1}{u_\tau^2} = \frac{v_0 x_2}{\nu} + \frac{1}{2}\left(\frac{v_0 x_2}{\nu}\right)^2 + \ldots, \tag{14}$$

or

$$\frac{\overline{U}_1}{u_\tau} = \frac{u_\tau x_2}{\nu} + \frac{1}{2}\left(\frac{v_0}{u_\tau}\right)\left(\frac{u_\tau x_2}{\nu}\right)^2 + \ldots. \tag{15}$$

It is clear that for all cases in which at the outer edge of the sublayer $v_0 x_2/\nu \ll 1$, the quadratic terms in (14) and (15) may be neglected. Sublayer similarity according to (13) then has no advantage over similarity according to the law of the wall for boundary layers on impervious surfaces (1). Experimental evidence (to be discussed presently) has shown that only for boundary layers at 'moderate' suction rates ($0{\cdot}04 < -v_0/u_\tau < 0{\cdot}10$ approximately) the viscous sublayer is sufficiently thick to justify the use of (13) as a similarity law. In this context it should be noticed that (13) becomes trivial for $v_0 \to 0$, so that the limit law of the wall in any case is unlikely to be suitable at very small suction or blowing rates.

Velocity profiles obtained in some experiments with sucked turbulent boundary layers have been plotted in figure 1. All layers concerned are asymptotic layers in zero pressure gradient. For these layers the momentum thickness is constant, so that the momentum integral equation reduces to $v_0 U_0 = -u_\tau^2$ (see § 3). This provides a relatively easy way to obtain an accurate value of u_τ. Since the state of equilibrium of a boundary layer is supposed to have no effect on the flow in the inner layer (Clauser 1956), the results presented here are supposed to possess a wider validity than for asymptotic layers only. The velocity profiles in figure 1 show the following main characteristics. First, the viscous sublayers of two asymptotic layers on a smooth sintered-bronze surface (Kay 1948) are rather thick, extending well beyond $-v_0 x_2/\nu = 1$ and coinciding with the theoretical curve according to (12). For these layers the limit law of the wall, (13), is apparently a suitable similarity law. Second, it should be noticed that the velocity profiles in figure 1 exhibit a more or less clearly distinguishable semi-logarithmic region; the slope of this region is approximately the same for all layers concerned. This is considered to be an essential property of a similarity law; the experimental evidence therefore agrees with the proposed limit law of the wall.

The limit law of the wall has been investigated further by means of a series of experiments carried out by the author at the Aeronautical Engineering Department of the Technological University of Delft. For these experiments, a flat plate with a permeable surface (filter paper over narrow-mesh perforated sheet) was installed in the Department's Low Turbulence Wind Tunnel. Details of the experimental arrangement and of the data may be found in the author's dissertation (Tennekes 1964*b*). Fourteen boundary layers flowing along the porous surface were subjected to uniform suction in zero pressure gradient. Velocity traverses were made at 100 mm intervals in streamwise direction. The resulting velocity profiles were plotted according to the limit law of the wall (u_τ was deter-

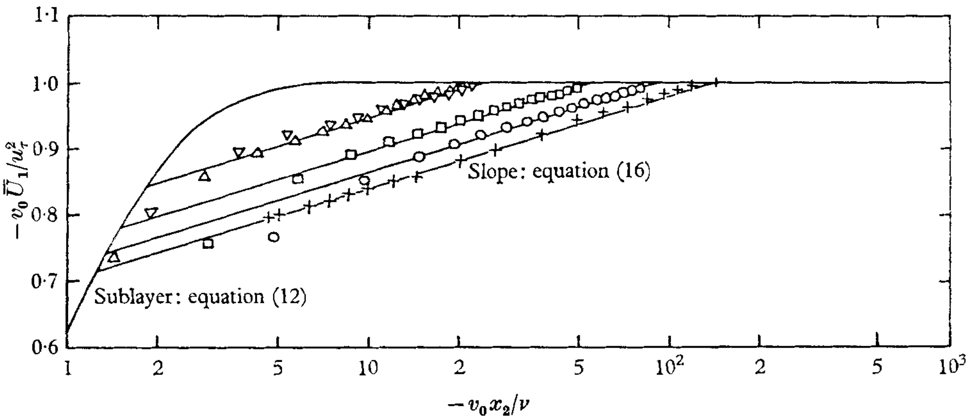

FIGURE 1. The limit law of the wall. Kay (1948): ▽, $-v_0/u_\tau = 0{\cdot}0548$; △, $-v_0/u_\tau = 0{\cdot}0576$. Dutton (1958): □, $-v_0/u_\tau = 0{\cdot}0665$; ○, $-v_0/u_\tau = 0{\cdot}0854$. Tennekes (1964*b*): +, $-v_0/u_\tau = 0{\cdot}0665$ (Run 2–429, $x_1 = 882$ mm; see also figure 5).

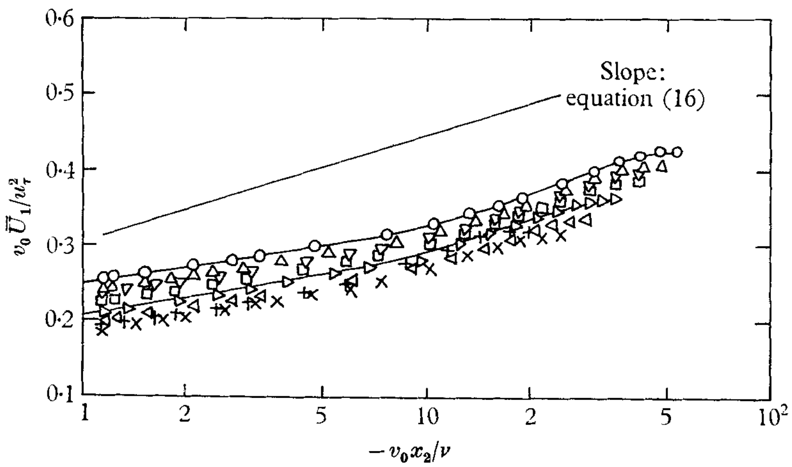

FIGURE 2. Velocity profiles in the co-ordinates of the limit law of the wall. Tennekes (1964*b*): Run 3–081; $-v_0/U_0 = 0{\cdot}00081$, $-v_0/\nu = 2{\cdot}87 \times 10^3\, m^{-1}$.

x_1 (mm)	$-100(v_0/u_\tau)$	x_1 (mm)	$-100(v_0/u_\tau)$
+ 163	1·61	□563	1·76
× 261	1·60	▽664	1·79
◁ 362	1·65	△763	1·81
▷463	1·71	○862	1·86

mined from the momentum integral equation); six representative cases are shown in figures 2 to 7. In these plots semi-logarithmic regions can be discerned; in most cases the slope of the semi-logarithmic region is equal to the slope of the profiles in figure 1. This evidence substantiates the validity of the limit law of the wall for boundary layers with sufficient suction. It is observed that the velocity profiles plotted in figures 2 and 7 exhibit a somewhat abnormal character. For the boundary layer presented in figure 2 the suction ratio $(-v_0/u_\tau)$ is too small for

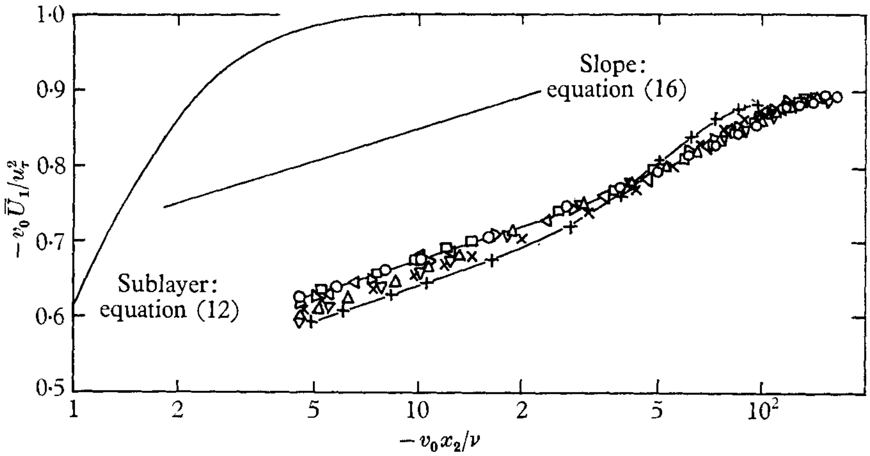

FIGURE 3. Velocity profiles in the co-ordinates of the limit law of the wall. Tennekes (1964*b*): Run 2–292; $-v_0/U_0 = 0{\cdot}00292$, $-v_0/\nu = 11{\cdot}3\times10^3\,m^{-1}$.

x_1 (*mm*)	$-100(v_0/u_\tau)$	x_1 (*mm*)	$-100(v_0/u_\tau)$
+ 182	5·06	□584	5·09
× 280	5·05	▽685	5·10
◁ 380	5·06	△785	5·10
▷482	5·08	○882	5·11

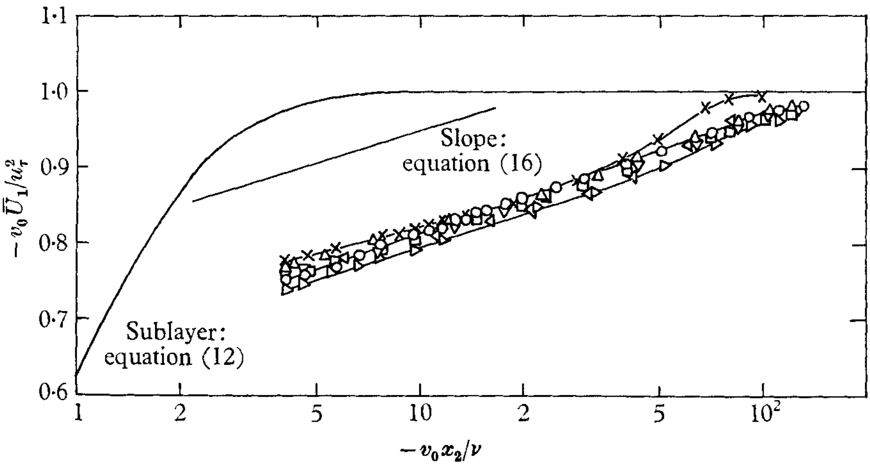

FIGURE 4. Velocity profiles in the co-ordinates of the limit law of the wall. Tennekes (1964*b*): Run 2–382; $-v_0/U_0 = 0{\cdot}00382$, $-v_0/\nu = 10{\cdot}1\times10^3\,m^{-1}$.

x_1 (mm)	$-100(v_0/u_\tau)$	x_1 (mm)	$-100(v_0/u_\tau)$
× 278	6·18	▽675	6·12
◁ 380	6·07	△776	6·13
▷480	6·07	○876	6·13
□577	6·10		

the limit law of the wall to be valid. For the layer in figure 7 the suction ratio is too large to maintain turbulent flow. This layer is expected to be in the process of reversal (backward 'transition') to laminar flow (cf. Favre, Dumas & Verollet 1961). The range of validity of the limit law of the wall will be discussed further in § 4.

The formal expression for the limit law of the wall (13) suggests that the appropriate velocity scale for turbulent boundary layers at moderate suction rates is proportional to u_τ^2/v_0. The experimental evidence presented in figures 1 to 7 shows that the logarithmic velocity scale is

$$w^* = -0{\cdot}06(u_\tau^2/v_0). \tag{16}$$

This relation is derived empirically by measuring the average slope of the semi-logarithmic region of the velocity profiles in figures 1 to 7. Straight lines conforming to (16) have been drawn in these figures to facilitate an appraisal of the agreement between the theory and the experimental data.

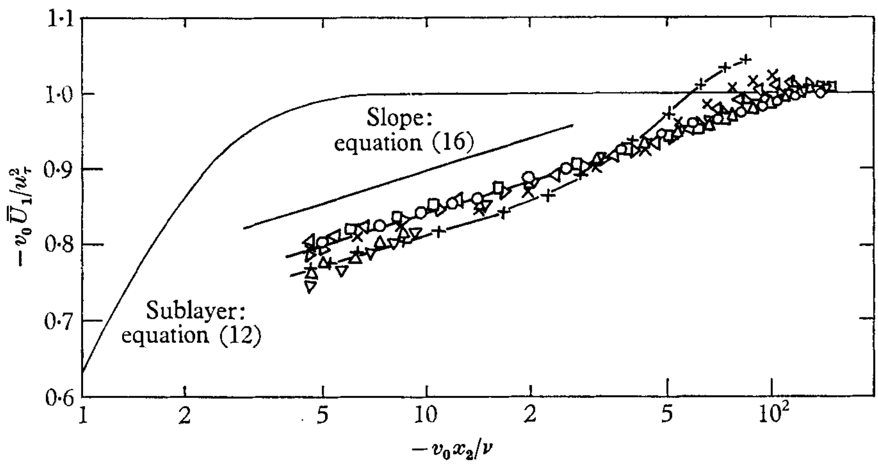

FIGURE 5. Velocity profiles in the co-ordinates of the limit law of the wall. Tennekes (1964*b*): Run 2–429; $-v_0/U_0 = 0{\cdot}00429, -v_0/\nu = 11{\cdot}5\times10^3\,m^{-1}$.

x_1 (*mm*)	$-100(v_0/u_\tau)$	x_1 (mm)	$-100(v_0/u_\tau)$
+ 181	6·70	□ 585	6·57
× 280	6·63	▽ 682	6·56
◁ 380	6·59	△ 785	6·55
▷ 482	6·58	○ 882	6·55

Evidence obtained in the course of the present investigation has shown that v_0/u_τ and kv_0/ν are parameters in the limit law of the wall. The complete formal expression for this law therefore should be written as

$$\frac{v_0\overline{U}_1}{u_\tau^2} = F\left(\frac{v_0x_2}{\nu}; \frac{v_0}{u_\tau}, \frac{kv_0}{\nu}\right). \tag{17}$$

The two parameters cause a parallel shift of the semi-logarithmic velocity profile. The effect of $-v_0/u_\tau$ is rather pronounced. For boundary-layer flow along smooth porous surfaces the following tentative formula describes the semi-logarithmic velocity profile (Tennekes 1964*b*)

$$-v_0\overline{U}_1/u_\tau^2 = 0{\cdot}06\ln(-v_0x_2/\nu) - 11(v_0/u_\tau) + A(kv_0/\nu), \tag{18}$$

where $A(kv_0/\nu)$ is a function, as yet undetermined, of the parameter kv_0/ν.

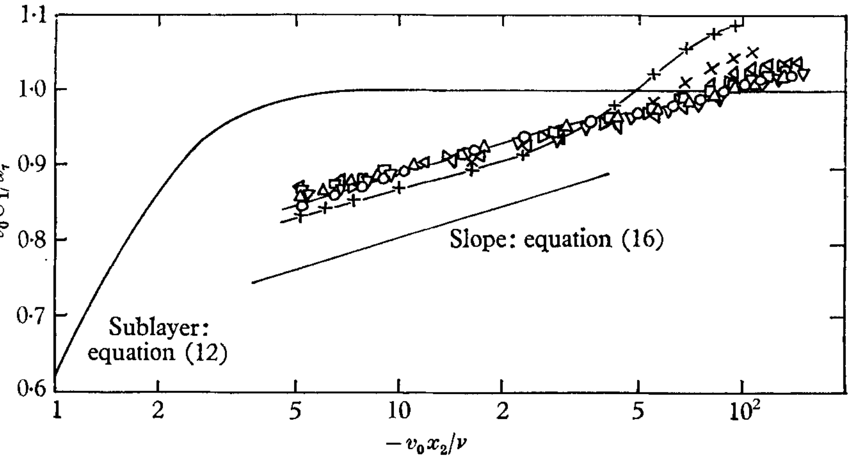

FIGURE 6. Velocity profiles in the co-ordinates of the limit law of the wall. Tennekes (1964*b*): Run 2–484; $-v_0/U_0 = 0{\cdot}00484$, $-v_0/\nu = 12{\cdot}9 \times 10^3\,\mathrm{m}^{-1}$.

x_1 (mm)	$-100(v_0/u_\tau)$	x_1 (mm)	$-100(v_0/u_\tau)$
+ 182	7·25	□ 581	7·06
× 282	7·15	▽ 679	7·05
◁ 382	7·10	△ 775	7·04
▷ 481	7·08	○ 880	7·03

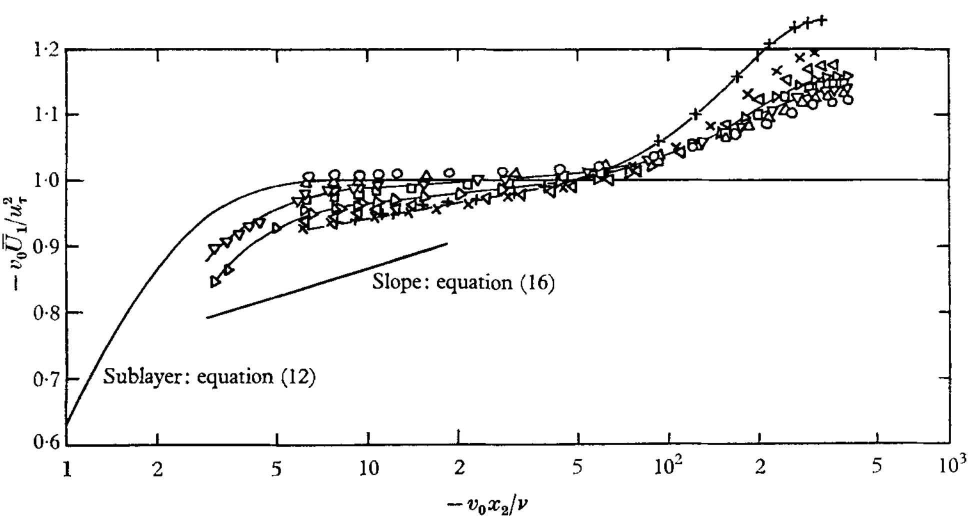

FIGURE 7. Velocity profiles in the co-ordinates of the limit law of the wall. Tennekes (1964*b*): Run 1–580; $-v_0/U_0 = 0{\cdot}00580$, $-v_0/\nu = 15{\cdot}5 \times 10^3\,\mathrm{m}^{-1}$.

x_1 (mm)	$-100(v_0/u_\tau)$	x_1 (mm)	$-100(v_0/u_\tau)$
+ 231	8·50	□ 631	8·16
× 332	8·31	▽ 730	8·14
◁ 435	8·25	△ 832	8·10
▷ 533	8·20	○ 930	8·08

3. The velocity defect law for turbulent boundary layers at moderate suction rates

With the information obtained in § 2 it is not a difficult task to formulate a velocity defect law for turbulent boundary layers with moderate suction. This 'limit velocity defect law' should have the same velocity scale as the limit law of the wall; in the region of overlap with the latter law the velocity profile should be semi-logarithmic. A region with these characteristics has been observed in most experimental velocity profiles, so that the following expression for the limit velocity defect law can be put forward with some confidence

$$v_0(\overline{U}_1 - U_0)/u_\tau^2 = G(x_2/\delta). \tag{19}$$

The scaling with U_0 and δ is characteristic for a deficiency law, cf. (2).

The limit velocity defect law will be valid only for boundary layers in equilibrium conditions (cf. Clauser 1956). A simple equilibrium layer, about which rather many experimental data are known, is the asymptotic layer in zero pressure gradient. For this layer, the growth due to skin friction is exactly compensated by suction, so that its thickness remains constant and all derivatives with respect to x_1 vanish. The remaining part of this section will be mainly devoted to a comparison of theory and experiments for asymptotic layers. No reliable experimental data exist about other sucked or blown equilibrium layers.

The relation between (19) and the equations of motion for asymptotic layers will now be investigated. This equation reads

$$v_0(d\overline{U}_1/dx_2) = d(-\overline{u_1 u_2})/dx_2. \tag{20}$$

This equation is valid for the inviscid flow in the outer layer only (viscous shear stress is negligible). Integration of (20) yields

$$v_0\,\overline{U}_1 - v_0\,U_0 = -\overline{u_1 u_2}. \tag{21}$$

The integration has been carried out from the outer edge of the layer inwards. It is not allowable to integrate from $x_2 = 0$ outwards since (20) is not valid in the direct vicinity of the wall. Now a state of equilibrium will be possible only if simultaneous similarity of the mean velocity profile and the Reynolds stress profile is compatible with the equations of motion. It seems suitable to non-dimensionalize the Reynolds stress $-\overline{u_1 u_2}$ by the value of the shear stress at the wall. Equation (21) can then be transformed into

$$v_0(\overline{U}_1 - U_0)/u_\tau^2 = -\overline{u_1 u_2}/u_\tau^2. \tag{22}$$

This equation clearly suggests that (19) is the proper velocity defect law for asymptotic layers.

In figure 8 the velocity profiles of some asymptotic layers, including one measured by the author, are presented in a plot according to (19). These velocity profiles apparently can be represented by a single curve, so that asymptotic layers are indeed equilibrium layers, which exhibit similarity of velocity profiles in the appropriate defect law plot. These velocity profiles apparently do not depend on any parameter (not counting the flow in the inner layer), in particular they do not depend on the suction ratio $-v_0/u_\tau$ and the skin friction coefficient $c_f = 2(u_\tau/U_0)^2$. The state of equilibrium of these layers may thus be called 'universal' (cf. Clauser 1956). The velocity profiles of these asymptotic layers exhibit extended semi-logarithmic parts, as may be observed in figure 8. The slope of the logarithm is equal to the logarithmic velocity scale given by (16), so that the hypothetical concept of related similarity laws for the inner and the outer layer, with a common velocity scale and semi-logarithmic region of overlap is verified experimentally for boundary layers at moderate suction rates. Further experimental evidence about non-asymptotic equilibrium layers would be welcome.

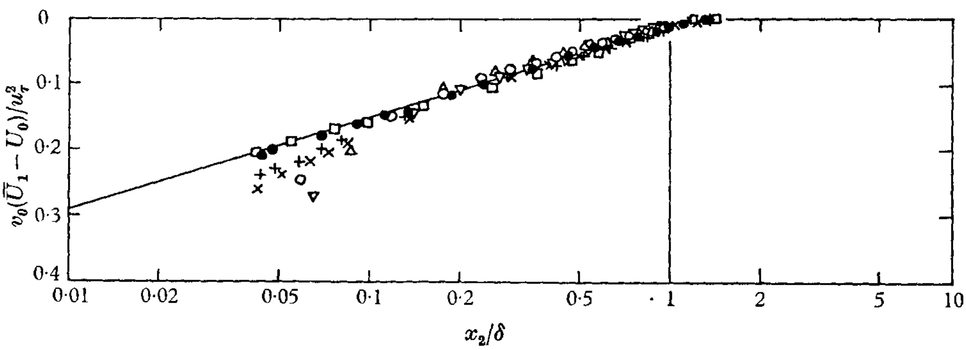

FIGURE 8. The limit velocity defect law: velocity profiles of asymptotic layers. Kay (1948): △, $-v_0/U_0 = 0{\cdot}00300$; ▽, $-v_0/U_0 = 0{\cdot}00332$. Dutton (1958): ○, $-v_0/U_0 = 0{\cdot}00443$, resp. 0·00730. Tennekes (1964*b*): Run 2–429: ●, +, × and □, $-v_0/U_0 = 0{\cdot}00429$.

x_1 (mm)	$-\Lambda$	x_1 (mm)	$-\Lambda$
●882	1·001	×682	1·004
+785	1·002	□585	1·007

It has been indicated in § 2 that the limit law of the wall is not valid at other than moderate suction rates, since at small suction rates the (empirically determined) logarithmic velocity scale differs from the one given in (16). However, a velocity scale according to (16), i.e. proportional to u_τ^2/v_0, is characteristic for asymptotic layers. This claim is supported by (22), which does not tolerate a different velocity scale if simultaneous similarity of Reynolds stress and mean velocity is to be achieved. This means that for asymptotic layers at small suction rates the velocity scales for the inner and the outer layers would be different. Within the present theory, this must be considered impossible. It is therefore concluded that at suction rates smaller than $-v_0/u_\tau = 0{\cdot}04$ approximately, no asymptotic layers can exist.† This conclusion is clearly tentative, awaiting further experimental and/or theoretical confirmation.

The limit velocity defect law (19) is supposed to be valid for all turbulent equilibrium layers at moderate suction rates, not only for asymptotic layers. In the velocity defect law for an arbitrary equilibrium layer one or more para-

† In relation to this feature, the similarity laws for boundary layers at moderate suction rates have been termed 'limit' similarity laws.

meters will occur which characterize the state of equilibrium and which are constant in equilibrium flow (Clauser 1956; Rotta 1962). For boundary layers on impervious surfaces the pressure gradient parameter Π has been introduced, (13). This parameter occurs in the momentum integral equation for boundary-layer flow. This equation reads, in non-dimensional form, including mass transfer (cf. Black & Sarnecki 1958)

$$\frac{\rho}{\tau_0}\frac{d(\theta U_0^2)}{dx_1} = 1+\Pi+\Lambda. \tag{23}$$

In this equation, the mass transfer parameter Λ is defined by

$$\Lambda = \rho v_0 U_0/\tau_0 = v_0 U_0/u_\tau^2. \tag{24}$$

If Π and Λ are constant, the relative contributions of skin friction, pressure gradient and suction or injection to the growth of the momentum deficiency θU_0^2 of a boundary layer are constant. With this in view, noting also that the use of Π as equilibrium parameter has received experimental support (Clauser 1956), Π and Λ appear to be a proper set of equilibrium parameters for sucked or blown boundary layers. This hypothesis can be checked in the case of asymptotic layers. For these layers, which are clearly equilibrium layers, each of the two parameters Π and Λ should have a unique value. This is indeed true, since asymptotic layers develop in a zero pressure gradient so that $\Pi = 0$; and for these layers θU_0^2 is constant so that $\Lambda = -1$ by virtue of (23). The considerations given above lead to the following tentative expression for the limit velocity defect law of an arbitrary equilibrium layer at a moderate suction rate

$$v_0(\overline{U}_1 - U_0)/u_\tau^2 = G(x_2/\delta;\Pi,\Lambda). \tag{25}$$

4. The logarithmic velocity scale

It has been shown that for turbulent boundary layers at moderate suction rates the concept of two similarity laws which are related by a common velocity scale and a semi-logarithmic velocity profile in a region of overlap is a sound one. In this section, the range of applicability of this concept will be enlarged through a further study of the logarithmic velocity scale.

A formal way to arrive at the logarithmic velocity scale at moderate suction rates will first be presented. The analysis runs parallel to the one which has been given in § 1, leading to the definition of w^*. Differentiation of the limit law of the wall, (13), or (17), and of the limit velocity defect law, (19), or (25), with respect to x_2 yields

$$\frac{v_0 x_2}{u_\tau^2}\frac{\partial \overline{U}_1}{\partial x_2} = \frac{v_0 x_2}{\nu}\frac{\partial F}{\partial(v_0 x_2/\nu)}, \tag{26}$$

$$\frac{v_0 x_2}{u_\tau^2}\frac{\partial \overline{U}_1}{\partial x_2} = \frac{x_2}{\delta}\frac{\partial G}{\partial(x_2/\delta)}. \tag{27}$$

The expressions on the right-hand side of (26) and (27) are formally independent of each other, so that they must be constant in order to be equal in the region of overlap. This means that in the region of overlap between the two limit similarity laws the velocity profile is semi-logarithmic and that the slope of this region is given by

$$w^* = x_2(\partial \overline{U}_1/\partial x_2) = C u_\tau^2/v_0. \tag{28}$$

The constant C is a 'universal' constant since the limit similarity laws (17) and (25) have no parameters in common. Experimental evidence supports (28) and indicates that $C = -0{\cdot}06$; cf. (16). It is observed that the semi-logarithmic velocity profiles observed in most experimental data are supported by the theory. For boundary layers with suction or blowing outside the range

$$0{\cdot}04 < -v_0/u_\tau < 0{\cdot}10$$

no appropriate similarity laws have so far been found. It is possible to formulate those laws by an empirical extension of the concept of a logarithmic velocity scale. For each experimentally determined velocity profile which has a semi-logarithmic region, the magnitude of w^* can be evaluated. It then remains to be determined on which variables w^* depends. For a solution of this problem, the logarithmic velocity scale is non-dimensionalized using u_τ. Equation (10) shows that for boundary layers on impervious surfaces

$$w^*/u_\tau = 2{\cdot}3. \tag{29}$$

Equation (16) shows that within the range of validity of the limit similarity laws

$$w^*/u_\tau = -0{\cdot}06(u_\tau/v_0). \tag{30}$$

These equations suggest that w^*/u_τ is a function of v_0/u_τ only. The logarithmic velocity scale, which is based on the relation between two similarity laws, can depend only on the variables that occur in both laws. A wall law is characteristically independent of U_0 and its derivatives and of δ; a defect law does not depend on the surface roughness k. These considerations also imply that w^*/u_τ depends on v_0/u_τ only.

In figures 9 and 10 the empirically-determined logarithmic velocity scales of a large number of sucked and blown turbulent boundary layers have been collected. Also plotted is the hyperbola given by (30). It is observed that these data support the claim that w^*/u_τ is a function of v_0/u_τ only. Over a great range of values of the suction ratio $-v_0/u_\tau$, including all cases of blown boundary layers, the relation between w^*/u_τ and v_0/u_τ is very well approximated by

$$w^*/u_\tau = 2{\cdot}3(1+9v_0/u_\tau). \tag{31}$$

This linear relationship, which has also been plotted in figures 9 and 10, provides a sharp contrast with the hyperbolic relationship (30) associated with the limit similarity laws. So far, no satisfactory theoretical explanation has been found for (31).

The experimental data show that (30) cannot be used for $v_0/u_\tau > -0{\cdot}04$ and that (31) cannot be used for $v_0/u_\tau < -0{\cdot}08$. In the range $0{\cdot}04 < -v_0/u_\tau < 0{\cdot}08$ both curves represent the data reasonably well; however, (30) is preferable, since it is associated with the theoretically-supported limit similarity laws. At suction ratios larger than $-v_0/u_\tau = 0{\cdot}10$ approximately, most sucked boundary layers revert to laminar flow (cf. Favre *et al.* 1961) or attain such small values of the Reynolds number that the turbulence in the layer is no longer 'fully developed' (a necessary condition for the existence of separate similarity laws).

Therefore, the range of values of v_0/u_τ at which the limit similarity laws may be used has been restricted to the so-called 'moderate' suction rates: $0{\cdot}04 < -v_0/u_\tau < 0{\cdot}10$.

It has been indicated that (31) is valid also for blown turbulent boundary layers. For these layers, Mickley & Smith (1963) derived a velocity scale based on the maximum shear stress within the boundary layer. They observed that the velocity profiles of blown boundary layers, if non-dimensionalized using u_τ^* (the

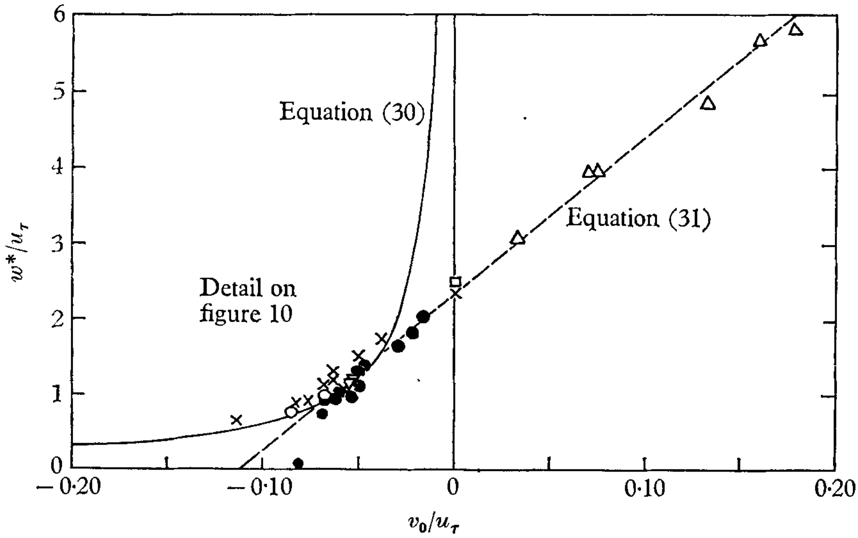

FIGURE 9. The logarithmic velocity scale. Data: ▽, Kay (1948); □, Clauser (1956); △, Mickley & Davis (1957); ○, Dutton (1958); ×, Black & Sarnecki (1958); ●, Tennekes (1964*b*).

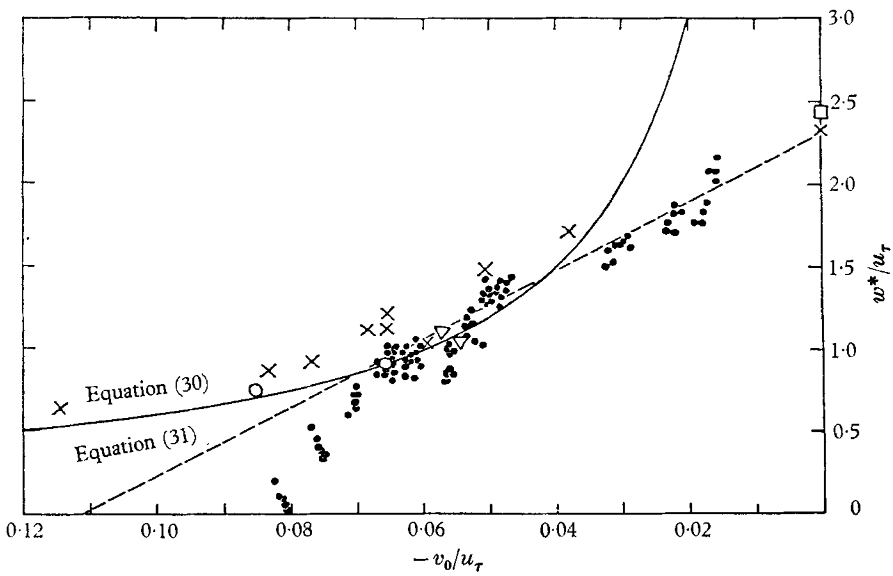

FIGURE 10. Detail of figure 9.

square root of the maximum shear stress in the layer), exhibit a semi-logarithmic region with the same slope as in the case of boundary layers on impervious surfaces. This property is analogous to the main property of scaling with w^*, i.e. that the slope of the logarithmic velocity profile is unity, whatever the value of v_0/u_τ. It is concluded that w^* and u_τ^* are proportional to each other; more specifically that

$$w^* = 2{\cdot}3u_\tau^*. \tag{32}$$

This relation provides a physical interpretation of the meaning of w^* for blown turbulent boundary layers and, using (31), also shows the way in which u_τ^* depends on v_0 and u_τ (this relation was not given by Mickley & Smith). It should be noticed that in most sucked boundary layers the shear stress does not attain an extremum, so that for negative values of v_0/u_τ the modified friction velocity u_τ^* cannot be defined following the procedure of Mickley & Smith. However, the relation between w^* and u_τ^* at least indicates that the logarithmic velocity scale is related in some manner to the characteristic level of shear stress in the region of overlap between the inner and the outer layer.

The logarithmic velocity scale w^* may be used to formulate a more general set of similarity laws

$$\frac{\overline{U}_1}{w^*} = \phi\left(\frac{x_2 u_\tau^2}{\nu w^*}; \frac{v_0}{u_\tau}, \frac{k u_\tau^2}{\nu w^*}\right), \tag{33}$$

$$(\overline{U}_1 - U_0)/w^* = \psi(x_2/\delta; \Pi, \Lambda). \tag{34}$$

These equations will be called the 'normalized' similarity laws. By substitution of (29), these laws transform into those for turbulent boundary layers on impervious surfaces; by substitution of (30), the limit similarity laws reappear. The normalized similarity laws are designed so that the slope of the semi-logarithmic velocity profile is unity, whatever the value of v_0/u_τ (this is the only parameter on which the slope depends). It is therefore clear that by varying any of the parameters occurring in (33) and (34) the velocity profile can only shift parallel to itself. This property should make experimental determination of the effects of the various parameters relatively easy. At present, the number of data available in the literature is insufficient to deduce the required relations with any accuracy.

To conclude this section, it should be noted that the concept of a logarithmic velocity scale may have a far wider range of applicability than sucked and blown turbulent boundary layers only. For all turbulent boundary layers which exhibit distinct inner and outer layers the logarithmic velocity scale may be determined experimentally. Experimental data and/or theory will then show the nature of the dependence of w^* on variables like heat transfer, compressibility and magnetic fields.

This paper is a condensed version of a thesis with the same title submitted to the Technological University of Delft for the degree of Doctor of Technical Sciences.

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